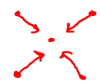
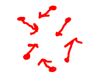
Tuesday, March 24, 2015
6:57 PM

A few key ideas related to black holes:

- Singularity theorems: You may be under the impression that perfect collapse to a point (Schwarzchild) or ring (Kerr) singularity is a feature of the high degree of symmetry assumed, e.g.



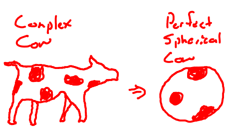
, and would probably not occur in realistic (perturbed) cases.



There a set of theorems due to Hawking and Penrose that suggest otherwise. They essentially use the notion of a "trapped surface" that forms during collapse, but before a singularity has formed. These are similar to event horizons (though technically distinct) and force the motion of the collapsing matter inside of them to decreasing r. The important part is that even though the physics at singularities cannot be described by GR, the trapped surfaces (and event horizons for that matter) are, and so GR predicts its own shortcoming!

- Cosmic Censorship Conjecture: This is an unproven (but well supported) idea that any singularity that results from collapse will always be hidden behind an event horizon. Some motivation for this comes from the singularity theorems themselves. This does not completely preclude the existence of an uncloaked (naked) singularity since it only applies to collapse. We will see an important exception.

- No-hair theorem: Stationary, asymptotically flat black holes are completely characterized by mass (M), charge (Q) and angular momentum (J).



If you think about it, this is pretty amazing. It says that all of the complexity in a very macroscopic system is essentially lost if it collapses. This will pose a significant conundrum a little later in this lecture.

- Area theorem: Hawking showed that assuming the weak energy condition (essentially $\rho \geq 0$) that the area of an event horizon can never decrease. For Schwarzchild this is obvious since $M_{BH} \to M_{BH} + \delta M \Rightarrow r_{horizon} = 2GM_{BH} \to 2GM_{BH} + 2G\delta M$.

- Energy and Momentum: While we have a pretty good idea of how to identify conserved quantities for test particles (use Killing vectors and 4-momentum) it isn't quite clear how we define conserved quantities for a geometry, e.g. Minkowski, Schwarzchild, Kerr. This is a subtle topic and there are a couple of approaches useful in different situations.

If we have a Killing vector field for a geometry we can use Komar integrals to define conserved quantities. If we use the $K^\mu = (1,0,0,0)$ for t-independence, we find that for $\mathbb{M}^4$: $E = 0$, S: $E = M$, K: $E = M$. Using $R^\mu = (0,0,0,1)$ for Kerr we find J as expected. So this means that with this definition, all of the "conserved" energy of a BH is described by M.

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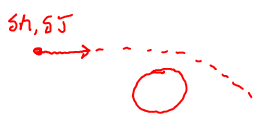
Back to the Kerr geometry. This provides highly nontrivial tests and implications for the preceding results.

Recall that for "over-extreme" Kerr BHs, i.e. $a^2 > G^2M^2$, there would exist a naked singularity. Could this actually happen? In binary systems accretion disks can swell the angular momentum of a BH to very near G^2M^2. Assume for a moment that the BH actually reaches being extremal, i.e. $a^2 = G^2M^2$. Then all we would have to do is feed it a bit of mass that has $\delta J > \delta M$ (which is certainly possible). This looks like it would push the BH to over extreme, hence into a naked singularity, hence violating the C.C.C.

But not so fast. If one actually studies the motion of such a bit of mass in the extremal Kerr geometry, it turns out that it always scatters, i.e. is not absorbed by the BH.



To increase δJ keeping δM fixed we either need to jack $\vec{v}$ or $r_\perp$.

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The Kerr geometry has yet another new and important feature.

Consider an object at rest outside of the BH (using thrusters or some other means).

Then: $u^\mu = (\underset{t}{u^0}, \underset{r}{0}, \underset{\theta}{0}, \underset{\phi}{0})$

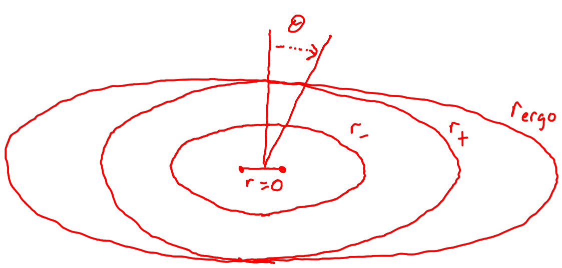
But: $u_\mu u^\mu = g_{00} u^0 u^0 = -\left(1 - \frac{2GMar}{\rho^2}\right) u^{0\,2} \overset{\text{always!}}{=} -1$

What this implies is that for $\left(1 - \frac{2GMar}{\rho^2}\right) < 1$, no object can remain at rest. But this includes a region outside of the BH! Note: Inside of the BH not being able to remain at rest is expected.

Recall: $\rho^2 = r^2 + a^2\cos^2\theta \Rightarrow \left(1 - \frac{2GMar}{r^2 + a^2\cos^2\theta}\right) < 1 \Rightarrow r < r_{ergo}(\theta) = GM + \sqrt{G^2M^2 - a^2\cos^2\theta}$

compare to

$r_+ = GM + \sqrt{G^2M^2 - a^2}$



This effect is called "frame-dragging"

There is another important aspect of the ergosphere region:

Consider $T^\mu = (1, 0, 0, 0) \Rightarrow T_\mu P^\mu = -m_0\left(1 - \frac{2GMar}{\rho^2}\right)\frac{dt}{d\tau} - \frac{m_0 2GMar}{\rho^2}\frac{d\phi}{d\tau} = -E_0$

↖ Killing vector for t-independence

Sign is chosen so that for $r \to \infty$
$E \to m_0 \frac{dt}{d\tau} > 0$

Same sign!

Then: $E_0 = m_0 \underbrace{\left(1 - \frac{2GMar}{\rho^2}\right)}_{\substack{>0 \;\; r > r_{ergo} \\ <0 \;\; r < r_{ergo}}} \underbrace{\frac{dt}{d\tau}}_{>0} + \underbrace{\frac{m_0 2GMar}{\rho^2}\frac{d\phi}{d\tau}}_{>0}$

For $\begin{cases} r > r_{ergo} & E_0 > 0 \\ r < r_{ergo} & E_0 \text{ can be positive or negative} \end{cases}$

This means that if an object is inside of the ergosphere and has $E_0 < 0$, then it can't escape.

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Consider an object outside of r_{ergo} w/ $E_0 > 0$. It enters the ergosphere and splits into $\underset{>0}{E_0} = \underset{>0}{E_1} + \underset{<0}{E_2}$ $\Rightarrow E_1 > E_0 > 0$ so E_1 can leave the ergosphere, but the E_2 is trapped.

To an outside observer an object w/ E_0 approaches the BH, delivers negative energy E_2 (or siphons off energy $-E_2$) and then leaves. Now the energy E_2 includes contributions from the rest mass and motion of the object, but once it is absorbed and included in the energy of the BH, as discussed earlier, it must now appear as a contribution to the mass of BH. In other words $E_2 = \delta M_{BH}$. But since $E_2 < 0$, we are reducing the mass of the BH.

Now to get this process to work, i.e. to have E_2 be absorbed and E_1 be on a trajectory that leaves the ergosphere, one can show that $J_2 \leq \frac{E_2}{\Omega_H}$ where $\Omega_H = \frac{a}{r_+^2 + a^2}$ is the angular velocity of the horizon. Since $E_2 < 0$ this tells us that J_2 and Ω_H have opposite signs, hence the absorbed object must have angular momentum opposite in direction of the BH. But this means that absorption of 2 reduces the angular momentum of the BH by $\delta J_{BH} = J_2$.

One important thing to remember is that this process requires an ergosphere region. Therefore the limit of this is when $J_{BH} = 0$ or Kerr $\rightarrow$ Schwarzschild or $r_{ergo} \rightarrow r_+ = 2GM$ beyond which nothing escapes.

For a perfect Penrose process $J_2 = \frac{E_2}{\Omega_H}$ or $\delta J = \frac{\delta M}{\Omega_H}$ (changes for BH, henceforth just $\delta J, \delta M$)

So we start with M_{Kerr} and reduce it w/ Penrose process to $M_{Schwarzschild}$. Shouldn't the horizon area get smaller? Not so fast.

First of all, since $r_+ = GM + \sqrt{G^2M^2 - a^2}$ where $J = Ma$. From this, since both M and J are decreasing, it isn't immediately obvious what is happening to r_+. However we really should be considering the horizon area which is nontrivially related to r_+ by the geometry.

$A_H = \int \sqrt{\det \gamma}\, d\theta\, d\phi$ γ_{ij} - induced metric on horizon from ds^2 w/ $r = r_+$, $dr = dt = 0$

$$= 4\pi(r_+^2 + a^2)$$
$$= 8\pi G^2M^2 + 8\pi\sqrt{G^4M^4 - G^2\underbrace{M^2a^2}_{J^2}}$$

If we vary δM and δJ we find:

$$\delta A_H = \underbrace{\frac{8\pi G a}{\Omega_H \sqrt{G^2M^2 - a^2}}}_{>0} \underbrace{(\delta M - \Omega_H \delta J)}_{\text{recall that } \delta J \leq \frac{\delta M}{\Omega_H}}$$

So $\delta A_H \geq 0$ and the area theorem is (nontrivially!) obeyed.

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The absolute nondecreasing of A_H sounds a bit like entropy. In some sense, we should expect a BH to have some observable entropy (we can observe A_H) since:

Beckenstein: If a BH had no observable entropy we could take an external system w/ entropy S_0 and upon throwing it into the BH, decrease the entropy of the observable universe, thus violating the 2nd law of thermodynamics.

To preserve the 2nd law, BHs must admit an observable entropy. One might think to use the mass M of the BH to correlate w/ entropy, but the Penrose process allows M to decrease in certain cases. It is only A_H which is a suitable proxy for entropy.

But this already tells us something deep. While entropy usually scales w/ the volume of a system, in this case it scales w/ the area. This points to the holographic nature of gravity, since information from 4D is captured in a 3D surface.

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Quantifying this idea, from the Kerr case:

$$\delta M = \frac{\kappa}{8\pi G}\,\delta A + \Omega_H\,\delta J \qquad \text{w/}\quad \kappa = \frac{\sqrt{G^2M^2 - a^2}}{2GM\left(GM + \sqrt{G^2M^2 - a^2}\right)}$$

κ is the "surface gravity" of the BH, or roughly, how strong the gravitational pull is near the horizon.

Comparing to $dE = TdS - PdV$ we would associate:

$$E = M$$
$$-PdV = \Omega_H\,\delta J$$
$$TdS = \frac{\kappa}{8\pi G}\,\delta A$$

It is tempting to identify $T = \frac{\kappa}{8\pi G}$ and $dS = \delta A$, but in truth the split isn't obvious.

Enter Hawking: Hawking considered QFT in the curved geometry near the horizon of a BH. Note: He was not doing quantum gravity (which is still not completely understood), he was doing perfectly well-defined QFT w/ minimal coupling.

Even if you don't trust QFT in curved space, you could make do w/ QFT in flat space and then apply the equivalence principle.

The Unruh effect is the consequence of uniform acceleration in flat space. A uniformly accelerated observer in flat space experiences not $\mathbb{M}^4$, but rather a Rindler spacetime. Quantizing a field in terms of Rindler time is very different than w/ Minkowski time (we use time to identify allowable positive frequency modes from $e^{i\omega t}$ dependence). The end result is that a uniformly accelerated observer in the vacuum (no particles) of $\mathbb{M}^4$ actually sees a thermal distribution of all allowed particle types (dominated by lowest mass) coming at them w/ the temperature $T = \frac{a}{2\pi}$ where a is the acceleration of the observer.

But now we use the equivalence principle to replace $a = \kappa$ and find $T = \frac{\kappa}{2\pi}$ and hence $dS = \frac{\delta A}{4G} \Rightarrow S = \frac{A}{4G}$

Black Hole Evaporation

Perhaps the most suprising aspect of Hawking's result is that BHs seem to radiate. The cartoon version of this is:

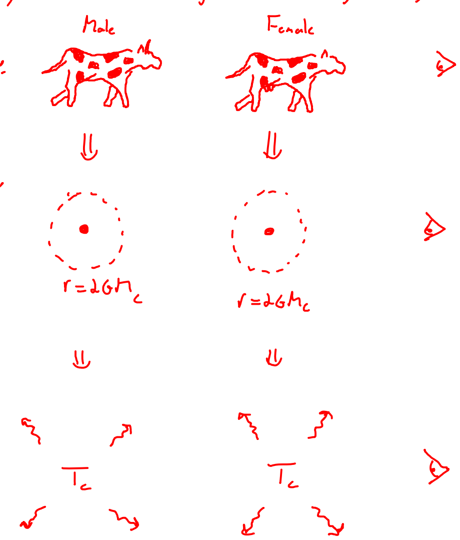
Why surprising? Conservation of energy implies that the BH is losing energy and hence mass in this process and (as easily seen for Schwarzchild) this means the horizon area is decreasing. Does this violate the area theorem? Actually no! The area theorem assumed a weak energy condition ($\sim \rho \geq 0$), but these quantum fluctuations can have $\rho < 0$, hence violating the w.e.c.

So BHs can evaporate! Of course for large BHs this rate is small especially compared to any accretion. However for $\underbrace{\text{microscopic BHs}}_{\text{threatened LHC}}$ it leads to them being very short-lived.

Black Hole Information Paradox

Perhaps the most perplexing thing about BH evaporation is the seeming loss of information. First note that in principle, in an otherwise empty universe and given a long enough time, any BH will completely evaporate.

Now consider 2 non-rotating, equal mass cows:



At this stage you could that information is "hidden" behind horizon.

Uh-oh... no more horizon, only T_c radiation which is identical!

Resolving this puzzle is almost certainly going to require a well-understood quantum theory of gravity!